

CW SIMPLICIAL RESOLUTIONS OF SPACES WITH AN APPLICATION TO LOOP SPACES

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ABSTRACT. We show how a certain type of CW simplicial resolutions of spaces by wedges of spheres may be constructed, and how such resolutions yield an obstruction theory for a given space to be a loop space.

1. INTRODUCTION

A simplicial resolution of a space \mathbf{X} by wedges of spheres is a simplicial space \mathbf{W}_\bullet such that (a) each space \mathbf{W}_n is homotopy equivalent to a wedge of spheres, and (b) for each $k \geq 1$, the augmented simplicial group $\pi_k \mathbf{W}_\bullet \rightarrow \pi_k \mathbf{X}$ is acyclic (see §3.5 below). Such resolutions, first constructed by Chris Stover in [St, §2], are dual to the “unstable Adams resolutions” of [BK, I, §2], and have a number of applications: see §3.10 below and [St, DKSS, DKS1, B1, B5, B6, B7].

However, the Stover construction yields very large resolutions, which do not lend themselves readily to computation, and no other construction was hitherto available. In particular, it was not clear whether one could find minimal resolutions of this type. The purpose of this note is to show that any space X has simplicial resolutions by wedges of spheres, which may be constructed from purely algebraic data, consisting of an (arbitrary) simplicial resolution of $\pi_* X$ as a Π -algebra – that is, as a graded group with an action on the primary homotopy operations on it (see §3.1 below):

Theorem A. *Every free simplicial Π -algebra resolution of a realizable Π -algebra $\pi_* X$ is realizable topologically as a simplicial resolution by wedges of spheres.*

and in fact such resolutions can be given a convenient “CW structure” (§3.15). There is an analogous result for maps (Theorem 3.24).

Since *no* such resolution of a non-realizable Π -algebra can be realized (see §3.16 below), this completely determines which free simplicial Π -algebra resolutions are realizable.

The Theorem implies that in the spectral sequences of [St, B1, DKSS] we can work with minimal resolutions, and allows us to identify the higher homotopy operations of [B5, B1, B7] as lying in appropriate cohomology groups (compare [B6, 4.17] and [B8, §6]). A generalization of Theorem A to other model categories appears in [B9].

As an application of such CW resolutions, we describe an obstruction theory for deciding whether a given space \mathbf{X} is a loop space, in terms of higher homotopy operations. One such theory was given in [B7], but the present approach does not require a given H -space structure on \mathbf{X} , and may be adapted also to the existence of A_n -structures (and thus subsumes [B6]). It is summarized in

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Theorem B. *A space \mathbf{X} with trivial Whitehead products is homotopy equivalent to a loop space if and only if the higher homotopy operations of §5.10 below vanish coherently.*

1.1. Notation and conventions. $\mathcal{G}p$ will denote the category of groups, \mathcal{T} that of topological spaces, and \mathcal{T}_* that of pointed topological spaces with base-point preserving maps. The full subcategory of 0-connected spaces will be denoted by $\mathcal{T}_c \subset \mathcal{T}_*$. The category of simplicial sets will be denoted by \mathcal{S} and that of pointed simplicial sets by \mathcal{S}_* ; we shall use boldface letters: $\mathbf{X}, \mathbf{S}^n, \dots$ to denote objects in any of these four categories. If $f : \mathbf{X} \rightarrow \mathbf{Y}$ is a map in one of these categories, we denote by $f_\# : \pi_* \mathbf{X} \rightarrow \pi_* \mathbf{Y}$ the induced map in the homotopy groups.

1.2. Organization. In section 2 we review some background on simplicial objects and bisimplicial groups, and in section 3 we recall some facts on Π -algebras, and prove our main results on CW resolutions of spaces by wedges of spheres: Theorem A (=Theorem 3.20) and Theorem 3.24. In section 4 we define a certain cosimplicial simplicial space up-to-homotopy, which can be rectified if and only if \mathbf{X} is a loop space. In section 5 we construct a certain collection of *face-codegeneracy polyhedra*, which are used to define the higher homotopy operations referred to in Theorem B (=Theorem 5.12). We also show how the theorem may be used in reverse to calculate a certain tertiary operation in $\pi_* \mathbf{S}^7$.

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2. SIMPLICIAL OBJECTS

We first provide some definitions and facts on simplicial objects:

2.1. Definition. Let Δ denote the category of ordered sequences $\mathbf{n} = \langle 0, 1, \dots, n \rangle$ ($n \in \mathbb{N}$), with order-preserving maps. A *simplicial object* over a category \mathcal{C} is a functor $X : \Delta^{op} \rightarrow \mathcal{C}$, usually written X_\bullet , which may be described explicitly as a sequence of objects $\{X_k\}_{k=0}^\infty$ in \mathcal{C} , equipped with face maps $d_i^k : X_k \rightarrow X_{k-1}$ and degeneracies $s_j^k : X_k \rightarrow X_{k+1}$ (usually written simply d_i, s_j , for $0 \leq i, j \leq k$), satisfying the usual simplicial identities ([Ma, §1.1]). If $I = (i_1, i_2, \dots, i_r)$ is some multi-index, we write d_I for $d_{i_1} \circ d_{i_2} \circ \dots \circ d_{i_r}$, with $d_\emptyset := id$; and similarly for s_I . An *augmented* simplicial object is one equipped with an augmentation $\varepsilon : X_0 \rightarrow Y$ (for $Y \in \mathcal{C}$), with $\varepsilon d_0 = \varepsilon d_1$.

The category of simplicial objects over \mathcal{C} is denoted by $s\mathcal{C}$. We write $s_{\langle n \rangle} \mathcal{C}$ for the category *n-simplicial objects* over \mathcal{C} (that is, objects of the form $\{X_k\}_{k=0}^n$, with the relevant face maps and degeneracies), and denote the truncation functor $s\mathcal{C} \rightarrow s_{\langle n \rangle} \mathcal{C}$ by τ_n .

For technical convenience in the next two sections we shall be working mainly in the category of simplicial groups, denoted by \mathcal{G} (rather than $s\mathcal{G}p$); objects in \mathcal{G} will be denoted by capital letters X, Y , and so on. A simplicial object $X_\bullet = (X_0, X_1, \dots)$ in $s\mathcal{G}$ is thus a bisimplicial group, which has an *external* simplicial dimension (the n in $X_n \in \mathcal{G}$), as well as the *internal* simplicial dimension k (inside \mathcal{G}), which we shall denote by $(X_n)_k^{int}$, if necessary.

2.2. Simplicial sets and groups. The standard n simplex in \mathcal{S} is denoted by $\Delta[n]$, generated by $\sigma_n \in \Delta[n]_n$. $\dot{\Delta}[n]$ denotes the sub-object of $\Delta[n]$ generated by $d_i\sigma_n$ ($0 \leq i \leq n$). The simplicial n -sphere is $\mathbf{S}^n := \Delta[n]/\dot{\Delta}[n]$, and the n -disk is $\mathbf{D}^n := C\mathbf{S}^{n-1}$.

Let $F : \mathcal{S} \rightarrow \mathcal{G}$ denote the (dimensionwise) free group functor of [Mi2, §2], and $G : \mathcal{S} \rightarrow \mathcal{G}$ be Kan's simplicial loop functor (cf. [Ma, Def. 26.3]), with $\bar{W} : \mathcal{G} \rightarrow \mathcal{S}$ the Eilenberg-Mac Lane classifying space functor (cf. [Ma, §21]). Recall that if $S : \mathcal{T} \rightarrow \mathcal{S}$ is the singular set functor and $\| - \| : \mathcal{S} \rightarrow \mathcal{T}$ the geometric realization functor (see [Ma, §1,14]), then the adjoint pairs of functors

$$(2.3) \quad \mathcal{T} \xrightleftharpoons[\|-\|]{S} \mathcal{S} \xrightleftharpoons[\bar{W}]{G} \mathcal{G}$$

induce isomorphisms of the corresponding homotopy categories (see [Q1, I, §5]), so that for the purposes of homotopy theory we can work in \mathcal{G} rather than \mathcal{T} .

2.4. Definition. In particular, $\mathbf{S}^n := F\mathbf{S}^{n-1} \in \mathcal{G}$ for $n \geq 1$ (and $\mathbf{S}^0 := G\mathbf{S}^0$ for $n = 0$) will be called the n -dimensional \mathcal{G} -sphere, in as much as $[\mathbf{S}^n, G\mathbf{X}]_{\mathcal{G}} \cong \pi_n \mathbf{X} = [\mathbf{S}^n, \mathbf{X}]$ for any Kan complex $\mathbf{X} \in \mathcal{S}$. Similarly, $\mathbf{D}^n := F\mathbf{D}^{n-1}$ will be called the n -dimensional \mathcal{G} -disk.

2.5. Definition. In any complete category \mathcal{C} , the *matching object* functor $M : \mathcal{S}^{op} \times s\mathcal{C} \rightarrow \mathcal{C}$, written $M_{\mathbf{A}}X_{\bullet}$ for a (finite) simplicial set $\mathbf{A} \in \mathcal{S}$ and $X_{\bullet} \in s\mathcal{C}$, is defined by requiring: (a) $M_{\Delta[n]}X_{\bullet} := X_n$, and (b) if $\mathbf{A} = \text{colim}_i \mathbf{A}_i$, then $M_{\mathbf{A}}X_{\bullet} = \lim_i M_{\mathbf{A}_i}X_{\bullet}$ (see [DKS2, §2.1]). In particular, if \mathbf{A}_n^k is the subcomplex of $\dot{\Delta}[n]$ generated by the last $(n - k + 1)$ faces $(d_k\sigma_n, \dots, d_n\sigma_n)$, we write $M_n^k X_{\bullet}$ for $M_{\mathbf{A}_n^k}X_{\bullet}$: explicitly,

$$(2.6) \quad M_n^k X_{\bullet} = \{(x_k, \dots, x_n) \in (X_{n-1})^{n+1} \mid d_i x_j = d_{j-1} x_i \text{ for all } k \leq i < j \leq n\}.$$

and the map $\delta_n^k : X_n \rightarrow M_n^k X_{\bullet}$ induced by the inclusion $\mathbf{A}_n^k \hookrightarrow \Delta[n]$ is defined $\delta_n^k(x) = (d_k x, \dots, d_n x)$. The original matching object of [BK, X, §4.5] was $M_n^0 X_{\bullet} = M_{\Delta[n]} X_{\bullet}$, which we shall further abbreviate to $M_n X_{\bullet}$; each face map $d_k : X_{n+1} \rightarrow X_n$ factors through $\delta_n := \delta_n^0$. See also [Hi, XVII, 87.17].

2.7. Remark. Note that for $X \in \mathcal{G}$ and $\mathbf{A} \in \mathcal{S}$ we have $M_{\mathbf{A}}X \cong \text{Hom}_{\mathcal{G}}(F\mathbf{A}, X) \in \mathcal{G}p$ (cf. §2.2), so for $X_{\bullet} \in s\mathcal{G}$ also $(M_{\mathbf{A}}X)_k \cong \text{Hom}_{\mathcal{G}}(F\mathbf{A}, (X_{\bullet})_k^{int})$ in each simplicial dimension k .

2.8. Definition. $X_{\bullet} \in s\mathcal{G}$ is called *fibrant* if each of the maps $\delta_n : X_n \rightarrow M_n X_{\bullet}$ ($n \geq 0$) is a fibration in \mathcal{G} (that is, a surjection onto the identity component – see [Q1, II, 3.8]). This is just the condition for fibrancy in the Reedy model category, (see [R]), as well as in that of [DKS1], but we shall not make explicit use of either.

By analogy with Moore's normalized chains (cf. [Ma, 17.3]) we have:

2.9. Definition. Given $X_{\bullet} \in s\mathcal{G}$, we define the n -cycles object of X_{\bullet} , written $Z_n X_{\bullet}$, to be the fiber of $\delta_n : X_n \rightarrow M_n X_{\bullet}$, so $Z_n X_{\bullet} = \{x \in X_n \mid d_i x = 0 \text{ for } i = 0, \dots, n\}$ (cf. [Q1, I, §2]). Of course, this definition really makes sense only when X_{\bullet} is fibrant

(§2.8). Similarly, the n -chains object of X_\bullet , written $C_n X_\bullet$, is defined to be the fiber of $\delta_n^1 : X_n \rightarrow M_n^1 X_\bullet$.

If $X_\bullet \in \mathcal{SG}$ is fibrant, the map $d_0^n = d_0|_{C_n X_\bullet} : C_n X_\bullet \rightarrow Z_{n-1} X_\bullet$ is the pullback of $\delta_n : X_n \rightarrow M_n X_\bullet$ along the inclusion $\iota : Z_{n-1} X_\bullet \rightarrow M_n X_\bullet$ (where $\iota(z) = (z, 0, \dots, 0)$), so d_0^n is a fibration (in \mathcal{G}), fitting into a fibration sequence

$$(2.10) \quad Z_n X_\bullet \xrightarrow{j_n} C_n X_\bullet \xrightarrow{d_0^n} Z_{n-1} X_\bullet.$$

2.11. Proposition. *For any fibrant $X_\bullet \in s\mathcal{C}$, the inclusion $\iota : C_n X_\bullet \hookrightarrow X_n$ induces an isomorphism $\iota_\star : \pi_* C_n X_\bullet \cong C_n(\pi_* X_\bullet)$ for each $n \geq 0$.*

Proof. (a) First note that if $j : \mathbf{A} \hookrightarrow \mathbf{B}$ is a trivial cofibration in \mathcal{S} , then $j^* : M_{\mathbf{B}} X_\bullet \rightarrow M_{\mathbf{A}} X_\bullet$ has a natural section $r : M_{\mathbf{A}} X_\bullet \rightarrow M_{\mathbf{B}} X_\bullet$ (with $j^* \circ r = id$) for any $X_\bullet \in \mathcal{SG}$: This is because by remark 2.7, $(M_{\mathbf{A}} X_\bullet)_k \cong \text{Hom}_{\mathcal{G}}(F\mathbf{A}, (X_\bullet)_k^{int})$ for $\mathbf{A} \in \mathcal{S}$; since $F\mathbf{A}$ is fibrant in \mathcal{G} , we can choose a left inverse $\rho : F\mathbf{B} \rightarrow F\mathbf{A}$ for $Fj : F\mathbf{A} \hookrightarrow F\mathbf{B}$, so $j^* : (M_{\mathbf{B}} X_\bullet)_k^{int} \rightarrow (M_{\mathbf{A}} X_\bullet)_k^{int}$ has a right inverse ρ^* , which is natural in $(X_\bullet)_k^{int}$; so these maps ρ^* fit together to yield the required map r .

This need not be true in general if j is not a weak equivalence, as the example of $M_2^1 X_\bullet \rightarrow M_1^0 X_\bullet$ shows.

(b) Given $\eta \in C_n \pi_m X_\bullet$ represented by $h : \mathcal{S}^m \rightarrow X_n$ with $d_k h \sim 0$ ($1 \leq k \leq n$), consider the diagram:

$$\begin{array}{ccccc}
 \mathcal{S}^m & & & & \\
 \downarrow h & \searrow \sim 0 & & & \\
 X_n & \xrightarrow{\delta_n^{k+1}} & M_n^{k+1} X_\bullet & & \\
 \downarrow \delta_n^k & \searrow & \downarrow j^* = (d_k, \dots, d_k) & & \\
 M_n^k X_\bullet & \xrightarrow{\quad} & M_n^{k+1} X_\bullet & & \\
 \downarrow \pi_k & \searrow & \downarrow & & \\
 X_{n-1} & \xrightarrow{\delta_{n-1}^k} & M_{n-1}^k X_\bullet & & \\
 \downarrow d_k & \searrow & & & \\
 & & & &
 \end{array}$$

(PB)

in which j^* is a fibration by (a) if $k \geq 1$, so the lower left-hand square is in fact a homotopy pullback square (see [Mat, §1]). By descending induction on $1 \leq k \leq n-1$, (starting with $\delta_n^n = d_n$), we may assume $\delta_n^{k+1} \circ h : \mathcal{S}^m \rightarrow M_n^{k+1} X_\bullet$ is nullhomotopic in \mathcal{C} , as is $d_k \circ h$, so the induced pullback map $\delta_n^k \circ h : \mathcal{S}^m \rightarrow M_n^k X_\bullet$ is also nullhomotopic by the universal property. We conclude that $\delta_n^1 \circ h \sim 0$, and since $\delta_n^1 : X_n \rightarrow M_n^1 X_\bullet$ is a fibration by (a), we can choose $h : \mathcal{S}^m \rightarrow X_n$ so that $\delta_n^1 h = 0$. Thus h lifts to $C_n X_\bullet = \text{Fib}(\delta_n^1)$, and ι_\star is surjective.

(c) Finally, the long exact sequence in homotopy for the fibration sequence

$$C_n X_\bullet \xrightarrow{\iota} X_n \xrightarrow{\delta_n^1} M_n^1 X_\bullet$$

implies that $\iota_\# : \pi_* C_n X_\bullet \rightarrow \pi_* X_n$ is monic, so $\iota_\star : \pi_* C_n X_\bullet \rightarrow C_n(\pi_* X_\bullet)$ is, too. \square

2.12. Definition. The dual construction to that of §2.5 yields the colimit

$$L_n X_\bullet := \coprod_{0 \leq i \leq n-1} X_{n-1} / \sim,$$

where for any $x \in X_{n-2}$ and $0 \leq i \leq j \leq n-1$ we set $s_j x$ in the i -th copy of X_{n-1} equivalent under \sim to $s_i x$ in the $(j+1)$ -st copy of X_{n-1} . $L_n X_\bullet$ has sometimes been called the “ n -th latching object” of X_\bullet . The map $\sigma_n : L_n X_\bullet \rightarrow X_n$ is defined $\sigma_n x_{(i)} = s_i x$, where $x_{(i)}$ is in the i -th copy of X_{n-1} .

3. Π -ALGEBRAS AND RESOLUTIONS

In this section we recall some definitions and prove our main results on Π -algebras and resolutions:

3.1. Definition. A Π -algebra is a graded group $G_* = \{G_k\}_{k=1}^\infty$ (abelian in degrees > 1), together with an action on G_* of the primary homotopy operations (i.e., compositions and Whitehead products, including the “ π_1 -action” of G_1 on the higher G_n ’s, as in [W, X, §7]), satisfying the usual universal identities. See [B3, §2.1] for a more explicit description. These are algebraic models of the homotopy groups $\pi_* \mathbf{X}$ of a space (or Kan complex) \mathbf{X} , in the same way that an algebra over the Steenrod algebra models its cohomology ring. The category of Π -algebras is denoted by $\Pi\text{-Alg}$.

We say that a space (or Kan complex, or simplicial group) \mathbf{X} *realizes* an (abstract) Π -algebra G_* if there is an isomorphism of Π -algebras $G_* \cong \pi_* \mathbf{X}$. (There may be non-homotopy equivalent spaces realizing the same Π -algebra – cf. [B5, §7.18]). Similarly, an abstract morphism of Π -algebras $\phi : \pi_* \mathbf{X} \rightarrow \pi_* \mathbf{Y}$ (between realizable Π -algebras) is *realizable* if there is a map $f : \mathbf{X} \rightarrow \mathbf{Y}$ such that $\pi_* f = \phi$.

3.2. Definition. The *free Π -algebra generated* by a graded set $T_* = \{T_k\}_{k=1}^\infty$ is $\pi_* \mathbf{W}$, where $\mathbf{W} = \bigvee_{k=1}^\infty \bigvee_{\tau \in T_k} \mathbf{S}_{(\tau)}^k$ (and we identify $\tau \in T_k$ with the generator of $\pi_k \mathbf{W}$ representing the inclusion $\mathbf{S}_{(\tau)}^k \hookrightarrow \mathbf{W}$).

If we let $\mathcal{F} \subset \Pi\text{-Alg}$ denote the full subcategory of free Π -algebras, and Π the homotopy category of wedges of spheres (inside $ho\mathcal{T}_*$ or $ho\mathcal{S}_*$ – or equivalently, the homotopy category of coproducts of \mathcal{G} -spheres in $ho\mathcal{G}$), then the functor $\pi_* : \Pi \rightarrow \mathcal{F}$ is an equivalence of categories. Thus any Π -algebra morphism $\varphi : G_* \rightarrow H_*$ is realizable (uniquely, up to homotopy), if G_* and H_* are free Π -algebras (actually, only G_* need be free).

3.3. Definition. Let $T : \Pi\text{-Alg} \rightarrow \Pi\text{-Alg}$ be the “free Π -algebra” comonad (cf. [Mc, VI, §1]), defined $TG_* = \coprod_{k=1}^\infty \coprod_{g \in G_k} \pi_* \mathbf{S}_{(g)}^k$. The counit $\varepsilon = \varepsilon_{G_*} : TG_* \rightarrow G_*$ is defined by $\iota_{(g)}^k \mapsto g$ (where $\iota_{(g)}^k$ is the canonical generator of $\pi_* \mathbf{S}_{(g)}^k$), and the comultiplication $\vartheta = \vartheta_{G_*} : TG_* \hookrightarrow T^2 G_*$ is induced by the natural transformation $\bar{\vartheta} : id_{\mathcal{F}} \rightarrow T|_{\mathcal{F}}$ defined by $x_k \mapsto \iota_{(x_k)}^k$.

3.4. Definition. An *abelian* Π -algebra is one for which all Whitehead products vanish.

These are indeed the abelian objects of $\Pi\text{-Alg}$ – see [B3, §2]. In particular, if \mathbf{X} is an H -space, then $\pi_* \mathbf{X}$ is an abelian Π -algebra (cf. [W, X, (7.8)]).

3.5. Definition. A simplicial Π -algebra A_\bullet is called *free* if for each $n \geq 0$ there is a graded set $T_*^n \subseteq A_n$ such that A_n is the free Π -algebra generated by T_*^n (§3.2), and each degeneracy map $s_j : A_n \rightarrow A_{n+1}$ takes T_*^n to T_*^{n+1} .

A *free simplicial resolution* of a Π -algebra G_* is defined to be an augmented simplicial Π -algebra $A_\bullet \rightarrow G_*$, such that

- (i) A_\bullet is a free simplicial Π -algebra,
- (ii) in each degree $k \geq 1$, the homotopy groups of the simplicial group $(A_\bullet)_k$ vanish in dimensions $n \geq 1$, and the augmentation induces an isomorphism $\pi_0(A_\bullet)_k \cong G_k$.

Such resolutions always exist, for any Π -algebra G_* – see [Q1, II, §4], or the explicit construction in [B1, §4.3].

3.6. Definition. For any $X \in \mathcal{G}$, a simplicial object $\mathbf{W}_\bullet \in s\mathcal{G}$ equipped with an augmentation $\varepsilon : W_0 \rightarrow X$ is called a *resolution of X by spheres* if each \mathbf{W}_n is homotopy equivalent to a wedge of \mathcal{G} -spheres, and $\pi_* \mathbf{W}_\bullet \rightarrow \pi_* X$ is a free simplicial resolution of Π -algebras.

3.7. Example. One example of such a resolution by spheres is provided by Stover’s construction; we shall need a variant in \mathcal{G} (as in [B7, §5]), rather than the original version of [St, §2], in \mathcal{T}_* . (The argument from this point on would actually work equally well in \mathcal{T}_* ; but we have already chosen to work in \mathcal{G} , in order to facilitate the proof of Proposition 2.11).

Define a comonad $V : \mathcal{G} \rightarrow \mathcal{G}$ for $G \in \mathcal{G}$ by

$$(3.8) \quad VG = \coprod_{k=0}^{\infty} \coprod_{\phi \in \text{Hom}_{\mathcal{G}}(\mathcal{S}^k, G)} \mathcal{S}_\phi^k \cup \coprod_{k=0}^{\infty} \coprod_{\Phi \in \text{Hom}_{\mathcal{G}}(\mathcal{D}^{k+1}, G)} \mathcal{D}_\Phi^{k+1},$$

where \mathcal{D}_Φ^{k+1} , the \mathcal{G} -disc indexed by $\Phi : \mathcal{D}^{k+1} \rightarrow G$, is attached to \mathcal{S}_ϕ^k , the \mathcal{G} -sphere indexed by $\phi = \Phi|_{\partial \mathcal{D}^{k+1}}$, by identifying $\partial \mathcal{D}^{k+1} := F\partial \mathcal{D}^k$ with \mathcal{S}^k (see §2.4 above). The coproduct here is just the (dimensionwise) free product of groups; the counit $\varepsilon : VG \rightarrow G$ of the comonad V is “evaluation of indices”, and the comultiplication $\vartheta : VG \hookrightarrow V^2G$ is as in §3.3.

Now given $X \in \mathcal{G}$, define $Q_\bullet \in s\mathcal{G}$ by setting $Q_n = V^{n+1}X$, with face and degeneracy maps induced by the counit and comultiplication respectively (cf. [Go, App., §3]). The counit also induces an augmentation $\varepsilon : Q_\bullet \rightarrow X$; and this is in fact a resolution of X by spheres (see [St, Prop. 2.6]).

3.9. Remark. Note that we need not use the \mathcal{G} -sphere and disk \mathcal{S}^k and \mathcal{D}^k of §2.4 in this construction; we can replace it by any other homotopy equivalent cofibrant pair of simplicial groups, so in particular by $(F\hat{\mathbf{D}}^k, F\hat{\mathbf{S}}^{k-1})$ for any pair of simplicial sets $(\hat{\mathbf{D}}^k, \hat{\mathbf{S}}^{k-1}) \simeq (\mathbf{D}^k, \mathbf{S}^{k-1})$.

3.10. The Quillen spectral sequence. A resolution by spheres $\mathbf{W}_\bullet \rightarrow X$ is in fact a resolution (i.e., cofibrant replacement) for the constant simplicial object $cX_\bullet \in s\mathcal{G}$ (i.e., $c(X)_n = X$, $d_i = s_j = id_X$) in an appropriate model category structure on $s\mathcal{G}$ – see [DKS1] and [B9]. However, we shall not need this fact; for our purposes it

suffices to recall that for any bisimplicial group $\mathbf{W}_\bullet \in s\mathcal{G}$, there is a first quadrant spectral sequence with

$$(3.11) \quad E_{s,t}^2 = \pi_s(\pi_t \mathbf{W}_\bullet) \Rightarrow \pi_{s+t} \text{diag } \mathbf{W}_\bullet$$

converging to the diagonal $\text{diag } \mathbf{W}_\bullet \in \mathcal{G}$, defined $(\text{diag } \mathbf{W}_\bullet)_k = (\mathbf{W}_k)_k^{\text{int}}$ (see [Q2]). Thus if $\mathbf{W}_\bullet \rightarrow X$ is a resolution by spheres, the spectral sequence collapses, and the natural map $\mathbf{W}_0 \rightarrow \text{diag } \mathbf{W}_\bullet$ induces an isomorphism $\pi_* \mathbf{X} \cong \pi_*(\text{diag } \mathbf{W}_\bullet)$. Combined with the fact that $\pi_* \mathbf{W}_\bullet$ is a resolution (in $s\Pi\text{-Alg}$) of $\pi_* \mathbf{X}$, this simple result has many applications – see for example [B1], [DKSS], and [St].

3.12. Definition. A *CW complex* over a pointed category \mathcal{C} is a simplicial object $R_\bullet \in s\mathcal{C}$, together with a sequence of objects \bar{R}_n ($n = 0, 1, \dots$) such that $R_n \cong \bar{R}_n \amalg L_n R_\bullet$ (§2.5), and $d_i^n|_{\bar{R}_n} = 0$ for $1 \leq i \leq n$. The objects $(\bar{R}_n)_{n=0}^\infty$ are called a *CW basis* for R_\bullet , and $\bar{d}_0^n := d_0|_{\bar{R}_n}$ is called the n -th attaching map for R_\bullet .

One may then describe R_\bullet explicitly in terms of its CW basis by

$$(3.13) \quad R_n \cong \coprod_{0 \leq \lambda \leq n} \coprod_{I \in \mathcal{J}_{\lambda,n}} \bar{R}_{n-\lambda}$$

where $\mathcal{J}_{\lambda,n}$ is the set of sequences I of λ non-negative integers $i_1 < i_2 < \dots < i_\lambda$ ($< n$), with $s_I = s_{i_\lambda} \circ \dots \circ s_{i_0}$ the corresponding λ -fold degeneracy (if $\lambda = 0$, $s_I = \text{id}$). See [B2, 5.2.1] and [Ma, p. 95(i)].

Such CW bases are convenient to work with in many situations; but they are most useful when each basis object is *free*, in an appropriate sense. In particular, if $\mathcal{C} = \Pi\text{-Alg}$, we have the following

3.14. Definition. A *CW resolution* of a Π -algebra G_* is a CW complex $A_\bullet \in s\Pi\text{-Alg}$, with CW basis $(\bar{A}_n)_{n=0}^\infty$ and attaching maps $\bar{d}_0^n : \bar{A}_n \rightarrow Z_{n-1} A_\bullet$, such that each \bar{A}_n is a free Π -algebra, and each attaching map $\bar{d}_0^n|_{C_n A_\bullet}$ is onto $Z_{n-1} A_\bullet$ (for $n \geq 0$, where we let \bar{d}_0^0 denote the augmentation $\varepsilon : A_\bullet \rightarrow G_*$ and $Z_{-1} A_\bullet := G_*$). Compare [B2, §5].

Every Π -algebra has a CW resolution (§3.14), as was shown in [B1, 4.4]: for example, one could take the graded set of generators \bar{T}_*^n for \bar{A}_n to be equal to the graded set $\pi_* Z_{n-1} A_\bullet$.

3.15. Definition. $Q_\bullet \in s\mathcal{G}$ is called a *CW resolution by spheres* of $X \in \mathcal{G}$ if $Q_\bullet \rightarrow X$ is a resolution by spheres (Def. 3.6), and Q_\bullet is a CW complex with CW basis $(\bar{Q}_n)_{n=0}^\infty$, such that each $\bar{Q}_n \in \mathcal{F}$ (i.e., \bar{Q}_n is homotopy equivalent to a wedge of spheres). The concept is defined analogously for $X \in \mathcal{S}$ or $X \in \mathcal{T}_*$.

3.16. Remark. Closely related to the problem of realizing abstract Π -algebras (§3.1) is that of realizing a free simplicial Π -algebra $A_\bullet \in s\Pi\text{-Alg}$: this is because, as noted in §3.5, every $G_* \in \Pi\text{-Alg}$ has a free simplicial resolution $A_\bullet \rightarrow G_*$; if it can be realized by a simplicial space $\mathbf{W}_\bullet \in s\mathcal{T}_c$ – or equivalently, via (2.3), by a bisimplicial space or group – then the spectral sequence (3.11) implies that $\pi_* \text{diag } \mathbf{W}_\bullet \cong G_*$. However, not every Π -algebra is realizable (see [B5, §8] or [B4, Prop. 4.3.6]).

It would nevertheless be very useful to know the converse: namely, that any free resolution of a *realizable* Π -algebra is itself realizable. This was mistakenly quoted as a

theorem in [B5, §6], where it was needed to make the obstruction theory for realizing Π -algebras described there of any practical use – and appeared as a conjecture in [B6, §4], in the context of an obstruction theory for a space to be an H -space.

In order to show that this conjecture is in fact true, we need several preliminary results:

3.17. Proposition. *Every CW resolution $A_\bullet \rightarrow \pi_* X$ of a realizable Π -algebra embeds in $\pi_* Q_\bullet$ for some resolution by spheres $Q_\bullet \rightarrow X$.*

Proof. To simplify the notation, we work here with topological spaces, rather than simplicial groups, changing back to \mathcal{G} if necessary via the adjoint pairs of §2.2.

Given a free simplicial Π -algebra resolution $A_\bullet \rightarrow J_*$ with CW basis $(\bar{A}_n)_{n=0}^\infty$, where $J_* = \pi_* \mathbf{X}$ for some $\mathbf{X} \in \mathcal{T}_*$, and \bar{A}_n is the free Π -algebra generated by the graded set T_*^n , let μ denote the cardinality of $\coprod_{n=0}^\infty \coprod_{k=0}^\infty T_k^n$, and set $\mathbf{X}' := \mathbf{X} \vee \bigvee_{n=0}^\infty \bigvee_{\lambda < \mu} \mathbf{D}^n$. Define new “spheres” and “disks” of the form $\hat{\mathbf{S}}^n := \mathbf{S}^n \vee \bigvee_{n=0}^\infty \bigvee_{\lambda < \mu} \mathbf{D}^n$ and $\hat{\mathbf{D}}^n := \hat{\mathbf{S}}^n \vee \mathbf{D}^n$. (This is to ensure that there will be at least μ different representatives for each homotopy class in $\pi_* \mathbf{X}'$ or $\pi_* \hat{\mathbf{S}}^n$.)

By remark §3.9 above, if we use the construction of §3.7 in \mathcal{T}_* (or in \mathcal{G} , *mutatis mutandis*) with these “spheres” and “disks”, and apply it to the space \mathbf{X}' , rather than to \mathbf{X} , we obtain a resolution by spheres $Q_\bullet \rightarrow \mathbf{X}'$.

We define $\phi : A_\bullet \hookrightarrow \pi_* Q_\bullet$ by induction on the simplicial dimension; it suffices to produce for each $n \geq 0$ an embedding $\bar{\phi}_n : \bar{A}_n \hookrightarrow C_n \pi_* Q_\bullet$ commuting with d_0 . If we denote $\varepsilon^A : A_0 \rightarrow \pi_* \mathbf{X} \cong \pi_* \mathbf{X}'$ by $\bar{d}_0^A : C_0 A_\bullet \rightarrow Z_{-1} A_\bullet =: A_{-1}$ and set $\phi_{-1} = id_{\pi_* X}$, then we may assume by induction we have a monomorphism $\phi_{n-1} : A_{n-1} \hookrightarrow \pi_* Q_{n-1}$ (taking generators to generators, and commuting with face and degeneracy maps).

For each Π -algebra generator ι_α in $(\bar{A}_n)_k$, if $d_0(\iota_\alpha) \neq 0$ then $\phi_{n-1}(d_0(\iota_\alpha)) \in Z_{n-1} \pi_k Q_\bullet$ is represented by some $g : \hat{\mathbf{S}}^k \rightarrow Q_{n-1}$, and we can choose distinct (though perhaps homotopic) maps g for different generators ι_α by our choice of $\hat{\mathbf{S}}^k$. Then by (3.8) there is a wedge summand $\hat{\mathbf{S}}_g^k$ in $Q_n = VQ_{n-1}$ (with no disks attached), and the corresponding free Π -algebra coproduct summand $\pi_* \hat{\mathbf{S}}_g^k$ in $\pi_* Q_n$, generated by ι_g , has $d_0(\iota_g) = [g] \in \pi_k Q_{n-1}$ and $d_i(\iota_g) = \iota_{d_{i-1}g} = 0 \in \pi_k Q_{n-1}$ for $1 \leq i \leq n$ by §3.7, since $[g] = \phi_{n-1}(d_0(\iota_\alpha)) \in Z_{n-1} \pi_k Q_\bullet$ and thus $d_i[g] = [d_i g] = 0$, and spheres indexed by nullhomotopic maps have disks attached to them. We see that $\iota_g \in C_n \pi_k Q_\bullet$, so we may define $\bar{\phi}_n(\iota_\alpha) = \iota_g$.

If $d_0(\iota_\alpha) = 0$, then all we need are enough distinct Π -algebra generators in $Z_n \pi_* Q_\bullet$: we cannot simply take ι_g for nullhomotopic $g : \mathbf{S}^k \rightarrow Q_{n-1}$, because of the attached disks; but we can proceed as follows:

Since $\hat{\mathbf{D}}^k = C\hat{\mathbf{S}}^k \vee \mathbf{D}^k$ and $\mathbf{X}' = \mathbf{X} \vee \bigvee_{i=0}^\infty \bigvee_{\lambda < \mu} \mathbf{D}^i$, we have μ distinct nonzero maps $F_\lambda : \hat{\mathbf{D}}^k \rightarrow \mathbf{X}'$ with $F_\lambda|_{C\hat{\mathbf{S}}^k} = *$. Define $H_+ = F_\lambda$, $H_- = *$; then $\mathbf{S}_H^k := \hat{\mathbf{D}}_{H_+}^k \cup_{\hat{\mathbf{S}}^{k-1}} \hat{\mathbf{D}}_{H_-}^k$ is, up to homotopy, a sphere wedge summand in Q_0 , and thus $\iota_{H_\lambda} \in \pi_k Q_0$ is a Π -algebra generator mapping to 0 under the augmentation. Similarly, define $\mathbf{S}_{G_\lambda}^k := \hat{\mathbf{D}}_{G_+}^k \cup_{\hat{\mathbf{S}}^{k-1}} \hat{\mathbf{D}}_{G_-}^k$ in Q_1 by $G_+ = *$, $G_- = * \perp \iota^k$ where ι^k is a homoeomorphism onto the summand \mathbf{D}^k in $\hat{\mathbf{D}}_{H_\lambda}^k$. Then $G_\lambda \sim *$ and $G_\lambda \neq *$ but $H \circ G = *$; thus ι_{H_λ} is a Π -algebra generator in $Z_1 \pi_k Q_\bullet$. By thus alternating the $+$ and $-$ we produce μ distinct Π -algebra generators in $Z_n \pi_* Q_\bullet$ for each n . \square

3.18. *Remark.* The referee has suggested an alternative proof of this Proposition, which may be easier to follow: rather than “fattening” the spheres and disks, we can modify the Stover construction of (3.8) by using μ copies of each sphere or disk for each $\phi \in \text{Hom}_{\mathcal{G}}(\mathcal{S}^k, G)$ or $\Phi \in \text{Hom}_{\mathcal{G}}(\mathcal{D}^{k+1}, G)$, respectively. The proof of [St, Prop. 2.6] still goes through, and so does the argument for embedding A_\bullet in $\pi_* Q_\bullet$ above.

3.19. Proposition. *Any free simplicial Π -algebra A_\bullet has a (free) CW basis $(\bar{A}_n)_{n=0}^\infty$.*

Proof. Start with $\bar{A}_0 = A_0$. For $n \geq 1$, assume $A_n = \coprod_{k=0}^\infty \coprod_{\tau \in T_k^n} \pi_* \mathcal{S}^k$. By Definition 3.5, $T_*^n \cong \bar{T}_*^n \cup \bigcup_{0 \leq \lambda \leq n} \bigcup_{I \in \mathcal{J}_{\lambda, n}} \hat{T}_*^{n-\lambda}$ (as in §3.13), so we can set $\hat{A}_n = \coprod_{k=0}^\infty \coprod_{\tau \in \hat{T}_k^n} \pi_* \mathcal{S}^k$; but $d_i|_{\hat{A}_n}$ need not vanish for $i \geq 1$.

However, given $\tau \in \hat{T}_k^n$, we may define $\tau_i \in (A_n)_k^{\text{int}}$ inductively, starting with $\tau_0 = \tau$, by $\tau_{i+1} = \tau_i s_{n-i-1} d_{n-i} \tau_i^{-1}$ (face and degeneracy maps taken in the external direction); we find that $\bar{\tau} := \tau_n$ is in $C_n A_\bullet$. If we define $\bar{\varphi} : \hat{T}_*^n \rightarrow A_n$ by $\bar{\varphi}(\tau) = \bar{\tau}$, by the universal property of free Π -algebras this extends to a map $\varphi : \hat{A}_n \rightarrow A_n$, which together with the inclusion $\sigma_n : L_n A_\bullet \hookrightarrow A_n$ yields a map $\psi : A_n \rightarrow A_n$ which is an isomorphism by the Hurewicz Theorem (cf. [B7, Lemma 2.5]). Thus we may set $\bar{A}_n := \varphi(\hat{A}_n)$, that is, the free Π -algebra generated by $\{\bar{\tau}\}_{\tau \in \hat{T}_*^n}$. Compare [K, §3]. \square

3.20. Theorem. *Every free simplicial Π -algebra resolution $A_\bullet \rightarrow \pi_* X$ of a realizable Π -algebra $\pi_* X$ is itself realizable by a CW resolution $R_\bullet \rightarrow X$ in $s\mathcal{G}$.*

Proof. By Propositions 3.17 and 3.19 we may assume A_\bullet has a (free) CW basis $(\bar{A}_n)_{n=0}^\infty$, and that there is a resolution by spheres $Q_\bullet \rightarrow X$ (in $s\mathcal{G}$) and an embedding of simplicial Π -algebras $\phi : A_\bullet \rightarrow Q_\bullet$. We may also assume that Q_\bullet is fibrant (§2.8), with $\varepsilon^Q : Q_0 \rightarrow X$ a fibration. We shall actually realize ϕ by a map of bisimplicial groups $f : R_\bullet \rightarrow Q_\bullet$.

Note that once R_\bullet has been defined through simplicial dimension n , for any $k \geq 0$ we have a commutative diagram

$$\begin{array}{ccccccccc}
 \pi_k C_n R_\bullet & \xrightarrow{(d_0)_\#} & \pi_k Z_{n-1} R_\bullet & \xrightarrow{(j_{n-1})_\#} & \pi_k C_{n-1} R_\bullet & \xrightarrow{(d_0)_\#} & \pi_k Z_{n-2} R_\bullet & \xrightarrow{(j_{n-2})_\#} & \pi_k C_{n-2} R_\bullet \\
 \downarrow \iota_* \cong & & \downarrow \rho_{n-1} & & \downarrow \iota_* \cong & & \downarrow \rho_{n-2} & & \downarrow \iota_* \cong \\
 C_n \pi_k R_\bullet & \xrightarrow{d_0^n} & Z_{n-1} \pi_k R_\bullet & \xrightarrow{\text{inc.}} & C_{n-1} \pi_k R_\bullet & \xrightarrow{d_0^{n-1}} & Z_{n-2} \pi_k R_\bullet & \xrightarrow{\text{inc.}} & C_{n-2} \pi_k R_\bullet
 \end{array}$$

(obtained by fitting together three of the long exact sequences of the fibrations (2.10)). The vertical maps are induced by the inclusions $C_n R_\bullet \hookrightarrow R_n$, and so on – see Proposition 2.11.

The only difficulty in constructing R_\bullet is that Proposition 2.11 does not hold for Z_n – i.e., the maps ρ_n in the above diagram in general need not be isomorphisms – so we may have an element in $Z_n A_\bullet$ represented by $\alpha \in C_n \pi_* R_\bullet = \pi_* C_n R_\bullet$ with $(d_0^n)_\#(\alpha) \neq 0$ (but of course $(j_{n-1})_\#(d_0^n)_\#(\alpha) = 0$). In this case we could not have $\beta \in \pi_* C_{n+1} R_\bullet = C_{n+1} A_\bullet$ with $(j_n)_\#(d_0^{n+1})_\#(\beta) = \alpha$, so $\pi_* R_\bullet$ would not be acyclic.

It is in order to avoid this difficulty that we need the embedding ϕ , since by definition this cannot happen for Q_\bullet : we know that $d_0^n : C_n \pi_* Q_\bullet \rightarrow Z_{n-1} \pi_* Q_\bullet$ is surjective for each $n > 0$, so $\rho_{n-1} : \pi_* Z_{n-1} Q_\bullet \rightarrow Z_{n-1} \pi_* Q_\bullet$ is, too, which implies that for each

$n > 0$:

$$(3.21) \quad \text{Im}\{(d_0^{n+1})_\# : \pi_* C_{n+1} Q_\bullet \rightarrow \pi_* Z_n Q_\bullet\} \cap \text{Ker}\{(j_n)_\# : \pi_* Z_n Q_\bullet \rightarrow \pi_* C_n Q_\bullet\} = 0$$

which we shall call *Property (3.21) for $Z_n Q_\bullet$* . (This implies in particular that $Z_n \pi_* Q_\bullet = \text{Ker}\{(d_0^n)_\# : \pi_* C_n Q_\bullet \rightarrow Z_{n-1} Q_\bullet\}$.)

Note that given any fibrant $K_\bullet \in s\mathcal{G}$ having Property (3.21) for $Z_m K_\bullet$ for each $0 < m \leq n$, if we consider the long exact sequence of the fibration $d_0^m : C_m K_\bullet \rightarrow Z_{m-1} K_\bullet$:

$$(3.22) \quad \dots \pi_{k+1} C_m K_\bullet \xrightarrow{(d_0^m)_\#} \pi_{k+1} Z_{m-1} K_\bullet \xrightarrow{\partial^{m-1}} \pi_k Z_m K_\bullet \xrightarrow{(j_{m-1})_\#} \pi_k C_{m-1} K_\bullet \dots,$$

we may deduce that

$$(3.23) \quad \partial^m|_{\text{Im}(\partial^{m-1})} \text{ is one-to-one, and surjects onto } \text{Im}(\partial^m)$$

for $0 < m \leq n$.

We now construct R_\bullet by induction on the simplicial dimension:

- (i) First, choose a fibration $\varepsilon^R : R_0 \rightarrow X$ realizing $\varepsilon^A : A_0 \rightarrow \pi_* X$. By §3.2, there is a map $f'_0 : R'_0 \rightarrow Q_0$ realizing ϕ_0 , so $\varepsilon^Q \circ f'_0 \sim \varepsilon^R$; since ε^Q is a fibration, we can change f'_0 to $f_0 : R_0 \rightarrow Q_0$ with $\varepsilon^Q \circ f_0 = \varepsilon^R$.
- (ii) Let $Z_0 R_\bullet$ denote the fiber of ε^R . Since $\varepsilon_\#^R = \varepsilon^A$ is a surjection, we have $\pi_* Z_0 R_\bullet = \text{Ker}(\varepsilon_\#^R) = Z_0 A_\bullet$, and d_0^A maps $C_1 A_\bullet$ onto $Z_0 A_\bullet$, so $\bar{d}_0^A : \bar{A}_1 \rightarrow A_0$ factors through $\pi_* Z_0 R_\bullet$, and we can thus realize it by a map $\bar{d}_0^R : \bar{R}_1 \rightarrow Z_0 R_\bullet$. Set $R'_1 := \bar{R}_1 \amalg L_1 R_\bullet$ (so $\pi_* R'_1 \cong A_1$), with $\delta'_1 : R'_1 \rightarrow M_1 R_\bullet = R_0 \times R_0$ equal to $(\bar{d}_0^R, 0) \perp \Delta$, and change δ'_1 to a fibration $\delta_1 : R_1 \rightarrow M_1 R_\bullet$. Again we can realize $\phi_1 : A_1 \rightarrow \pi_* Q_1$ by $f_1 : R_1 \rightarrow Q_1$ with $\delta_1^Q \circ f_1 = f_0 \circ \delta_1^R$, since δ_1^Q is a fibration; so we have defined $\tau_1 f : \tau_1 R_\bullet \rightarrow \tau_1 Q_\bullet$ realizing $\tau_1 \phi$.
- (iii) Now assume we have $\tau_n f : \tau_n R_\bullet \rightarrow \tau_n Q_\bullet$ realizing $\tau_n \phi$, with Property (3.21) holding for $Z_m R_\bullet$ for $0 < m < n$.

For each Π -algebra generator $\alpha \in \bar{A}_{n+1}$ (in degree k , say), (3.21) implies that $d_0^{n+1}(\alpha) \in \text{Ker}(d_0^n) = \text{Ker}((d_0^{R_n})_\#) \subset (C_n A_\bullet)_k = \pi_k C_n R_\bullet$, so by the exactness of (3.22) we can choose $\beta \in \pi_k Z_n R_\bullet$ such that $(j_n)_\# \beta = d_0^{n+1}(\alpha)$. This allows us to define $\bar{d}_0^R : \bar{R}_{n+1} \rightarrow Z_n R_\bullet$ so that $(j_n)_\# (\bar{d}_0^R)_\#$ realizes $(\text{inc.}) \circ \bar{d}_0^A : \bar{A}_{n+1} \rightarrow C_n A_\bullet$, as well as $\bar{f}_{n+1} : \bar{R}_{n+1} \rightarrow C_n Q_\bullet$ realizing $\phi_{n+1}|_{\bar{A}_{n+1}}$. Because $\bar{A}_{n+1} = \pi_* \bar{R}_{n+1}$ is a free Π -algebra, this implies the homotopy-commutativity of the outer rectangle in

$$\begin{array}{ccc} \bar{R}_{n+1} & \xrightarrow{\bar{f}_{n+1}} & C_{n+1} Q_\bullet \\ \bar{d}_0^R \downarrow & & \downarrow d_0^Q \\ Z_n R_\bullet & \xrightarrow{\quad Z_n f \quad} & Z_n Q_\bullet \\ j_n^R \downarrow & & \downarrow j_n^Q \\ C_n R_\bullet & \xrightarrow{C_n f} & C_n Q_\bullet \end{array}$$

(as well as the lower square, by the induction hypothesis). Thus $j_n^Q \circ Z_n f \circ \bar{d}_0^R \sim j_n^Q \circ d_0^Q \circ \bar{f}_{n+1}$, so $(j_n^Q)_\# \circ (Z_n f)_\# \circ (\bar{d}_0^R)_\# = (j_n^Q)_\# \circ (d_0^Q)_\# \circ (\bar{f}_{n+1})_\#$. By (3.21) this implies $(Z_n f)_\# \circ (\bar{d}_0^R)_\# = (d_0^Q)_\# \circ (\bar{f}_{n+1})_\#$, so (since $\pi_* \bar{R}_{n+1}$ is a free Π -algebra) also $Z_n f \circ \bar{d}_0^R \sim d_0^Q \circ \bar{f}_{n+1}$ – which means that we can choose \bar{f}_{n+1} so that $Z_n f \circ \bar{d}_0^R = d_0^Q \circ \bar{f}_{n+1}$ (since d_0^Q is a fibration). Thus if we set $\bar{\delta}_{n+1}^R : \bar{R}_{n+1} \rightarrow M_{n+1} R_\bullet$ to be $(\bar{d}_0^R, 0, \dots, 0)$, we have $M_{n+1} f \circ \bar{\delta}_{n+1}^R = \delta_{n+1}^Q \circ \bar{f}_{n+1}$.

If $\psi_{n+1}^R := \delta_{n+1}^R \circ \sigma_{n+1}^R$ (in the notation of §2.5 & 2.12) we set $R'_{n+1} := \bar{R}_{n+1} \amalg L_{n+1} R_\bullet$, and define $\delta'_{n+1} : R'_{n+1} \rightarrow M_{n+1} R_\bullet$, and $f'_{n+1} : R'_{n+1} \rightarrow Q_{n+1}$ respectively by $\delta'_{n+1} := (\bar{\delta}_{n+1}^R \amalg \psi_{n+1}^R)$ and $f'_{n+1} := (\bar{f}_{n+1} \amalg L_{n+1} f)$. We see that $(f'_{n+1})_\# = \phi_{n+1}$ and $M_{n+1} f \circ \delta'_{n+1} = \delta_{n+1}^Q \circ f'_{n+1}$, and this will still hold if we change δ'_{n+1} into a fibration, and extend f'_{n+1} to $f_{n+1} : R_{n+1} \rightarrow Q_{n+1}$. This defines $\tau_{n+1} f : \tau_{n+1} R_\bullet \rightarrow \tau_{n+1} Q_\bullet$ realizing $\tau_{n+1} \phi$.

- (iv) It remains to verify that $\tau_{n+1} R_\bullet$ so defined satisfies (3.21). However, (3.23) implies that we have a map of short exact sequences:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Im}(\partial_R^{n-1}) & \xrightarrow{\text{inc.}} & \pi_k Z_n R_\bullet & \longrightarrow & \text{Im}((j_n^R)_\#) \cong Z_n A_\bullet \longrightarrow 0 \\ & & \downarrow f_* & & \downarrow (Z_n f)_\# & & \downarrow Z_n \phi \\ 0 & \longrightarrow & \text{Im}(\partial_Q^{n-1}) & \xrightarrow{\text{inc.}} & \pi_k Z_n Q_\bullet & \longrightarrow & \text{Im}((j_n^Q)_\#) \cong Z_n \pi_k Q_\bullet \longrightarrow 0 \end{array}$$

in which the left vertical map is an isomorphism and the right map is one-to-one, so $(Z_n f)_\#$ is one-to-one, too. Therefore, $\text{Ker}((j_n^R)_\#) = \text{Ker}((j_n^Q)_\#) \cap \pi_* Z_n R_\bullet$, which implies that Property (3.21) holds for $Z_n R_\bullet$, too.

This completes the inductive construction of R_\bullet . \square

We also have an analogous result for maps:

3.24. Theorem. *If $K_\bullet \xrightarrow{\varepsilon^K} \pi_* X$ and $L_\bullet \xrightarrow{\varepsilon^L} \pi_* Y$ are two free simplicial Π -algebra resolutions, $g : X \rightarrow Y$ is a map in \mathcal{G} , and $\varphi : K_\bullet \rightarrow L_\bullet$ is a morphism of simplicial Π -algebras such that $\varepsilon^L \circ \varphi_0 = \pi_* g \circ \varepsilon^K$, then φ is realizable by a map $f : A_\bullet \rightarrow B_\bullet$ in $s\mathcal{G}$.*

Proof. Choose free CW bases for K_\bullet and L_\bullet , and realize the resulting CW resolutions by A_\bullet and B_\bullet respectively, where (as in the proof of Theorem 3.20) we may assume $d_0 : C_n B_\bullet \rightarrow Z_{n-1} B_\bullet$ is a fibration for each $n \geq 0$. $f_n : A_n \rightarrow B_n$ will be defined by induction on n : $\varphi_0 : K_0 \rightarrow L_0$ may be realized by a map $f'_0 : A_0 \rightarrow B_0$ (§3.2), and since ε^B is a fibration and $\varepsilon^B \circ f'_0 \sim g \circ \varepsilon^A$, we can choose a realization f_0 for φ_0 such that $\varepsilon^B \circ f_0 = g \circ \varepsilon^A$.

In general, $\bar{\varphi}_n = \varphi_n|_{\bar{K}_n} : \bar{K}_n \rightarrow C_n L_\bullet$ may be realized by a map $\bar{f}_n : \bar{A}_n \rightarrow C_n B_\bullet$ (Proposition 2.11), and since $d_0 : C_n B_\bullet \rightarrow Z_{n-1} B_\bullet$ is a fibration, we may choose \bar{f}_n so $d_0 \circ \bar{f}_n = Z_{n-1} f \circ d_0 : \bar{A}_n \rightarrow Z_{n-1} B_\bullet$. By induction this yields a map $f_n = L_n f \amalg \bar{f}_n : A_n = L_n A_\bullet \amalg \bar{A}_n \rightarrow L_n B_\bullet \amalg \bar{B}_n = B_n$ such that $\delta_n^B \circ f_n = M_n f \circ \delta_n^A : A_n \rightarrow M_n B_\bullet$, so f is indeed a simplicial morphism (realizing ϕ). \square

4. THE SIMPLICIAL BAR CONSTRUCTION

As an application of Theorem 3.20, we describe an obstruction theory for determining whether a given space X is, up to homotopy, a loop space (and thus a topological group – see [Mi1, §3]). In the next two sections we no longer need to work with simplicial groups, so we revert to the more familiar category of topological spaces; we can still utilize the results of the previous section via the adjoint pairs of (2.3).

4.1. Definition. A Δ -cosimplicial object E_Δ^\bullet over a category \mathcal{C} is a sequence of objects E^0, E^1, \dots , together with coface maps $d^i : E^n \rightarrow E^{n+1}$ for $1 \leq i \leq n$ satisfying $d^j d^i = d^i d^{j-1}$ for $i < j$ (cf. [RS]). Given an ordinary cosimplicial object E^\bullet (cf. [BK, X, 2.1]), we let E_Δ^\bullet denote the underlying Δ -cosimplicial object (obtained by forgetting the codegeneracies).

4.2. The cosimplicial James construction. Given a space $\mathbf{X} \in \mathcal{T}_*$, we define a Δ -cosimplicial space $\mathbf{U}_\Delta^\bullet = U(\mathbf{X})_\Delta^\bullet$ by setting $\mathbf{U}^n = \mathbf{X}^{n+1}$ (the Cartesian product), and $d^i(x_0, \dots, x_n) = (x_0, \dots, x_{i-1}, *, x_i, \dots, x_n)$. Note that $\text{colim } \mathbf{U}(X)_\Delta^\bullet \cong J\mathbf{X}$ (the James reduced product construction), and

4.3. Fact. If $\langle \mathbf{X}, m \rangle$ is a (strictly) associative H -space, we can extend $\mathbf{U}_\Delta^\bullet$ to a full cosimplicial space \mathbf{U}^\bullet by setting $s^j(x_0, \dots, x_n) = (x_0, \dots, m(x_j, x_{j-1}), \dots, x_n)$.

4.4. Definition. Let A_\bullet be a CW -resolution of the Π -algebra $\pi_* \mathbf{X} = \pi_* \mathbf{U}^0$, as in §3.14. We construct a Δ -cosimplicial augmented simplicial Π -algebra $(E_\bullet)_\Delta^\bullet \rightarrow \pi_* \mathbf{U}_\Delta^\bullet$, such that each E_\bullet^n is a CW -resolution of $\pi_* \mathbf{U}^n = \pi_*(\mathbf{X}^{n+1})$, with CW -basis $\{\bar{E}_r^n\}_{r=0}^\infty$. We start by setting $\bar{E}_r^0 = \bar{C}_r^0 = \bar{A}_r$ for all $r \geq 0$, and then define \bar{E}_r^n by a double induction (on $r \geq 0$ and then on $n \geq 0$) as

$$(4.5) \quad \bar{E}_r^n = \coprod_{0 \leq \lambda \leq n} \coprod_{I \in \mathcal{J}_{\lambda,n}} [\bar{C}_r^{n-\lambda}]_I,$$

where $\mathcal{J}_{\lambda,n}$ is as in (3.13) and $\bar{C}_r^m = 0 = \bar{C}_r^0$ for all $m, r \geq 0$.

The coface maps $d^i : E_r^{n-1} \rightarrow E_r^n$ are determined by the cosimplicial identities and the requirement that $d^i|_{[\bar{C}_r^{n-\lambda}]_{(i_1, \dots, i_n)}}$ be an isomorphism onto $[\bar{C}_r^{n-\lambda}]_{(i_1, \dots, i_n, i)}$ if $i > i_n$.

The only summand in (4.5) which is not defined is thus $[\bar{C}_r^n]_\emptyset$, which we denote simply by \bar{C}_r^n . We require that it be an n -th *cross-term* in the sense that $\bar{d}_0|_{\bar{C}_r^n}$ does not factor through the image of any coface map $d^i : E_{r-1}^{n-1} \rightarrow E_r^n$. Other than that, \bar{C}_r^n may be any free Π -algebra which ensures that (4.5) defines a CW -basis for a CW -resolution $E_\bullet^n \rightarrow \pi_* \mathbf{U}^n$. We shall call the double sequence $((\bar{C}_r^n)_{n=1}^\infty)_{r=1}^\infty$ a *cross-term basis* for $(E_\bullet)_\Delta^\bullet$.

Note that A_\bullet is a retract of E_\bullet^2 in two different ways (under the two coface maps d^0, d^1), corresponding to the fact that \mathbf{X} is a retract of $\mathbf{X} \times \mathbf{X}$ in two different ways; the presence of the cross-terms \bar{C}_r^2 indicates that $A_\bullet \times A_\bullet$ is a resolution of $\pi_* \mathbf{X}^2$, but not a free one, while $A_\bullet \amalg A_\bullet$ is a free simplicial Π -algebra, but not a resolution.

Similarly, $\mathbf{X} \times \mathbf{X}$ embeds in \mathbf{X}^3 in three different ways, and so on.

4.6. Example. For any $A_\bullet \rightarrow \pi_* \mathbf{X}$ we may set $\bar{C}_1^2 = \coprod_{S_x^p \hookrightarrow A_0^{(0)}} \coprod_{S_y^q \hookrightarrow A_0^{(1)}} S_{(x,y)}^{p+q-1}$, with $\bar{d}_0|_{S_{(x,y)}^{p+q-1}} = [\iota_x, \iota_y]$ (in the notation of §3.3). The higher cross-terms $\bar{C}_1^n = 0$ for $n \geq 3$, since any k -th order cross-term element z in $\coprod_{j=0}^n A_0^{(j)}$ ($k \geq 3$) is a sum of elements of the form $z = \zeta^\#[\dots [\iota_{(x_1)}^{r_1}, \iota_{(x_2)}^{r_2}], \iota_{(x_3)}^{r_3}], \dots, \iota_{(x_k)}^{r_k}]$, and then

$$z = d_0(\zeta^\#[\dots [\iota_{(x_1, x_2)}^{r_1+r_2-1}, s_0 \iota_{(x_3)}^{r_3}], \dots, s_0 \iota_{(x_k)}^{r_k}])).$$

4.7. Definition. Let ${}^h(\mathbf{W}_\bullet)_\Delta^\bullet \rightarrow \mathbf{U}_\Delta^\bullet$ be the Δ -cosimplicial augmented simplicial space up-to-homotopy which corresponds to $(E_\bullet)_\Delta^\bullet \rightarrow \pi_* \mathbf{U}_\Delta^\bullet$ via §3.2. Thus the various (co)simplicial morphisms exist, and satisfy the (co)simplicial identities, only in the homotopy category (we may choose representatives in \mathcal{T}_* , but then the identities are satisfied only up to homotopy). Each \mathbf{W}_r^n is homotopy equivalent to a wedge of spheres, and has a wedge summand $\bar{\mathbf{W}}_r^n \hookrightarrow \mathbf{W}_r^n$ corresponding to the CW -basis free Π -algebra summand $\bar{E}_r^n \hookrightarrow E_r^n$. We let \bar{C}_r^n denote the wedge summand of $\bar{\mathbf{W}}_r^n$ corresponding to $\bar{C}_r^n \hookrightarrow \bar{E}_r^n$.

4.8. Definition. An simplicial space $\mathbf{V}_\bullet \in s\mathcal{T}_*$ is called a *rectification* of a simplicial space up-to-homotopy ${}^h\mathbf{W}_\bullet$ if $\mathbf{V}_n \simeq \mathbf{W}_n$ for each $n \geq 0$, and the face and degeneracy maps of \mathbf{V}_\bullet are homotopic to the corresponding maps of ${}^h\mathbf{W}_\bullet$. See [DKS, §2.2], e.g., for a more precise definition; for our purposes all we require is that $\pi_* \mathbf{V}_\bullet$ be isomorphic (as a simplicial Π -algebra) to $\pi_*({}^h\mathbf{W}_\bullet)$. Similarly for rectification of (Δ) -cosimplicial objects, and so on.

By considering the proof of Theorem 3.20, we see that we can make the following

4.9. Assumption. $(E_\bullet)_\Delta^\bullet$ maps monomorphically into $\pi_* \mathbf{V}_\bullet(U_\Delta^\bullet)$, and ${}^h(\mathbf{W}_\bullet)_\Delta^\bullet \rightarrow \mathbf{U}_\Delta^\bullet$ can be rectified so as to yield a strict Δ -cosimplicial augmented simplicial space $(\mathbf{W}_\bullet)_\Delta^\bullet \rightarrow \mathbf{U}_\Delta^\bullet$ realizing $(E_\bullet)_\Delta^\bullet \rightarrow \pi_* \mathbf{U}_\Delta^\bullet$.

4.10. Definition. Now assume that $\pi_* \mathbf{X}$ is an abelian Π -algebra (Def. 3.4) – this is the necessary Π -algebra condition in order for \mathbf{X} to be an H -space – and let $\mu : \pi_* \mathbf{X} \times \pi_* \mathbf{X} \rightarrow \pi_* \mathbf{X}$ be the morphism of Π -algebras defined levelwise by the group operation (see [B6, §2]). This μ is of course associative, in the sense that $\mu \circ (\mu, id) = \mu \circ (id, \mu) : \pi_*(\mathbf{X}^3) \rightarrow \pi_* \mathbf{X}$, so it allows one to extend the Δ -cosimplicial Π -algebra $F_\Delta^\bullet := \pi_*(\mathbf{U}_\Delta^\bullet)$ to a full cosimplicial Π -algebra F^\bullet , defined as in §4.3.

Since $E_\bullet^n \rightarrow F^n = \pi_* \mathbf{U}^n$ is a free resolution of Π -algebras, the codegeneracy maps $s^j : F^n \rightarrow F^{n-1}$ induce maps of simplicial Π -algebras $s_\bullet^j : E_\bullet^n \rightarrow E_\bullet^{n-1}$, unique up to simplicial homotopy, by the universal property of resolutions (cf. [Q1, I, p. 1.14 & II, §2, Prop. 5]). Note, however, that the individual maps $s_r^j : E_r^n \rightarrow E_r^{n-1}$ are not unique, in general; in fact, different choices may correspond to different H -multiplications on \mathbf{X} .

These maps s^j make $(E_\bullet)_\Delta^\bullet \rightarrow F_\Delta^\bullet$ into a full cosimplicial augmented simplicial Π -algebra $E_\bullet^\bullet \rightarrow F^\bullet$, and thus ${}^h\mathbf{W}_\bullet^\bullet \rightarrow \mathbf{U}_\Delta^\bullet$ into a cosimplicial augmented simplicial space up-to-homotopy (for which we may assume by 4.9 that all simplicial identities, and all the cosimplicial identities involving only the coface maps, hold precisely).

4.11. Proposition. *The cosimplicial simplicial space up-to-homotopy ${}^h\mathbf{W}_\bullet^\bullet$ of §4.10 may be rectified if and only if \mathbf{X} is homotopy equivalent to a loop space.*

Proof. If \mathbf{X} is a loop space, it has a strictly associative H -multiplication $m : \mathbf{X} \times \mathbf{X} \rightarrow \mathbf{X}$ which induces μ on $\pi_*(-)$ (cf. [Gr, Prop. 9.9]), so $\mathbf{U}_\Delta^\bullet$ extends to a cosimplicial space \mathbf{U}^\bullet by Fact 4.3. Applying the functorial construction of [St, §2] to \mathbf{U}^\bullet yields a (strict) cosimplicial augmented simplicial space $(\mathbf{V}_\bullet)_\Delta^\bullet \rightarrow \mathbf{U}^\bullet$, and since we assumed $\pi_* \mathbf{W}_\bullet^n$ embeds in $\pi_* \mathbf{V}_\bullet^n$ for each n , ${}^h\mathbf{W}_\bullet^\bullet$ may also be rectified.

Conversely, if $\mathbf{W}_\bullet^\bullet$ is a (strict) cosimplicial simplicial space realizing E_\bullet^\bullet , then we may apply the realization functor for simplicial spaces in each cosimplicial dimension $n \geq 0$ to obtain $\|\mathbf{W}_\bullet^n\| \simeq \mathbf{U}^n = \mathbf{X}^{n+1}$ (by §3.10). The realization of the codegeneracy map $\|s^0\| : \|\mathbf{W}_\bullet^1\| \rightarrow \|\mathbf{W}_\bullet^0\|$ induces $\mu : \pi_*(\mathbf{X}^2) \rightarrow \pi_*\mathbf{X}$, so it corresponds to an H -space multiplication $m : \mathbf{X}^2 \rightarrow \mathbf{X}$ (see [B6, Prop. 2.7]).

The fact that $\|\mathbf{W}_\bullet^\bullet\|$ is a (strict) cosimplicial space means that all composite codegeneracy maps $\|s^0 \circ s^{j_1} \circ \dots \circ s^{j_{n-1}}\| : \|\mathbf{W}_\bullet^n\| \rightarrow \|\mathbf{W}_\bullet^0\|$ are equal, and thus all possible composite multiplications $\mathbf{X}^{n+1} \rightarrow \mathbf{X}$ (i.e., all possible bracketings in (2.6)) are homotopic, with homotopies between the homotopies, and so on – in other words, the H -space $\langle \mathbf{X}, m \rangle$ is an A_∞ space (see [St3, Def. 11.2]) – so that \mathbf{X} is homotopy equivalent to loop space by [St3, Theorem 11.4]. Note that we only required that the codegeneracies of ${}^h\mathbf{W}_\bullet^\bullet$ be rectified; after the fact this ensures that the full cosimplicial simplicial space is rectifiable. \square

In summary, the question of whether \mathbf{X} is a loop space reduces to the question of whether a certain diagram in the homotopy category, corresponding to a diagram of free Π -algebras, may be rectified – or equivalently, may be made ∞ -homotopy commutative.

5. POLYHEDRA AND HIGHER HOMOTOPY OPERATIONS

As in [B5, §4], there is a sequence of higher homotopy operations which serve as obstructions to such a rectification, and these may be described combinatorially in terms of certain polyhedra, as follows:

5.1. Definition. The N -permutohedron \mathbf{P}^N is defined to be the convex hull in \mathbb{R}^N of the points $p_\sigma = (\sigma(1), \sigma(2), \dots, \sigma(N))$, where σ ranges over all permutations $\sigma \in \Sigma_N$ (cf. [Z, §9]). It is $(N-1)$ -dimensional.

For any two integers $0 \leq n < N$, the corresponding (N, n) -face-codegeneracy polyhedron \mathbf{P}_n^N is a quotient of the N -permutohedron \mathbf{P}^N obtained by identifying two vertices p_σ and $p_{\sigma'}$ to a single vertex $\bar{p}_\sigma = \bar{p}_{\sigma'}$ of \mathbf{P}_n^N whenever $\sigma = (i, i+1)\sigma'$, where $(i, i+1)$ is an adjacent transposition and $\sigma(i), \sigma(i+1) > n$.

Since each facet A of \mathbf{P}^N is uniquely determined by its vertices (see below), the facets in the quotient \mathbf{P}_n^N are obtained by collapsing those of \mathbf{P}^N accordingly.

Note that \mathbf{P}_{N-1}^N is the N -permutohedron \mathbf{P}^N , and in fact the quotient map $q : \mathbf{P}^N \twoheadrightarrow \mathbf{P}_n^N$ is homotopic to a homeomorphism (though not a combinatorial isomorphism, of course) for $n \geq 1$. On the other hand, \mathbf{P}_0^N is a single point. For non-trivial examples of face-codegeneracy polyhedra, see Figures 1 & 2 below.

5.2. Fact. From the description of the facets of the permutohedron given in [GG], we see that \mathbf{P}_n^N has an edge connecting a vertex p_σ to any vertex of the form $p_{(i, i+1)\sigma}$ (unless $\sigma(i), \sigma(i+1) > n$, in which case the edge is degenerate).

More generally, let \bar{p}_σ be any vertex of \mathbf{P}_n^N . The facets of \mathbf{P}_n^N containing \bar{p}_σ are determined as follows:

Let $\mathbb{P} = \langle 1, 2, \dots, \ell_1 \mid \ell_1 + 1, \dots, \ell_2 \mid \dots \mid \ell_{i-1} + 1, \dots, \ell_i \mid \dots \mid \ell_{r-1} + 1, \dots, N \rangle$ be a partition of $1, \dots, N$ into r consecutive blocs, subject to the condition that for each $1 \leq j < r$ at least one of $\sigma(\ell_i)$, $\sigma(\ell_{i+1})$ is $\leq n$. Denote by n_i the number of j 's in the i -th bloc (i.e., $\ell_{i-1} + 1 \leq j \leq \ell_i$) such that $\sigma(j) \leq n$. Then \mathbf{P}_n^N will have a subpolyhedron $Q(\mathbb{P})$ (containing p_σ) which is isomorphic to the product

$$\mathbf{P}_{n_1}^{\ell_1} \times \mathbf{P}_{n_2}^{\ell_2 - \ell_1} \times \dots \times \mathbf{P}_{n_i}^{\ell_i - \ell_{i-1}} \times \dots \times \mathbf{P}_{n_r}^{N - \ell_{r-1}}.$$

This follows from the description of the facets of the N -permutohedron in [B5, §4.3].

We denote by $(\mathbf{P}_n^N)^{(k)}$ the union of all facets of \mathbf{P}_n^N of dimension $\leq k$. In particular, for $n \geq 1$ we have $\partial \mathbf{P}_n^N := (\mathbf{P}_n^N)^{(N-2)} = \mathbf{S}^{N-2}$, since the homeomorphism $\tilde{q} : \mathbf{P}^N \rightarrow \mathbf{P}_n^N$ preserves $\partial \mathbf{P}^N$.

5.3. Factorizations. Given a cosimplicial simplicial object E_\bullet as in §4.10, any composite face-codegeneracy map $\psi : E_{m+\ell}^{n+k} \rightarrow E_\ell^k$ has a (unique) canonical factorization of the form $\psi = \phi \circ \theta$, where $\theta : E_{m+\ell}^{n+k} \rightarrow E_{m+\ell}^k$ may be written $\theta = s^{j_1} \circ s^{j_2} \circ \dots \circ s^{j_n}$ for $0 \leq j_1 < j_2 < \dots < j_n < n+k$ and $\phi : E_{m+\ell}^k \rightarrow E_\ell^k$ may be written $\phi = d_{i_1} \circ d_{i_2} \circ \dots \circ d_{i_n}$ for $0 \leq i_1 < i_2 < \dots < i_n \leq m+\ell$.

Let $\mathcal{D}(\psi)$ denote the set of all possible (not necessarily canonical) factorizations of ψ as a composite of face and codegeneracy maps: $\psi = \lambda_{n+m} \circ \dots \circ \lambda_1$. We define recursively a bijective correspondence between $\mathcal{D}(\psi)$ and the vertices of an $(n+m)$ -permutohedron \mathbf{P}^{n+m} , as follows (compare [B5, Lemma 4.7]):

The canonical factorization $\psi = d_{i_1} \circ d_{i_2} \circ \dots \circ d_{i_n} \circ s^{j_1} \circ s^{j_2} \circ \dots \circ s^{j_n}$ corresponds to the vertex p_{id} . Next, assume that the factorization $\psi = \lambda_{n+m} \circ \dots \circ \lambda_1$ corresponds to p_σ . Then the factorization corresponding to $p_{\sigma'}$, for $\sigma = (i, i+1)\sigma'$, is obtained from $\psi = \lambda_1 \circ \dots \circ \lambda_{n+m}$ by switching λ_i and λ_{i+1} , using the identity $s^j \circ s^i = s^{i-1} \circ s^j$ for $i > j$ if λ_i and λ_{i+1} are both codegeneracies, and the identity $d_i \circ d_j = d_{j-1} \circ d_i$ for $i < j$ if they are both face maps.

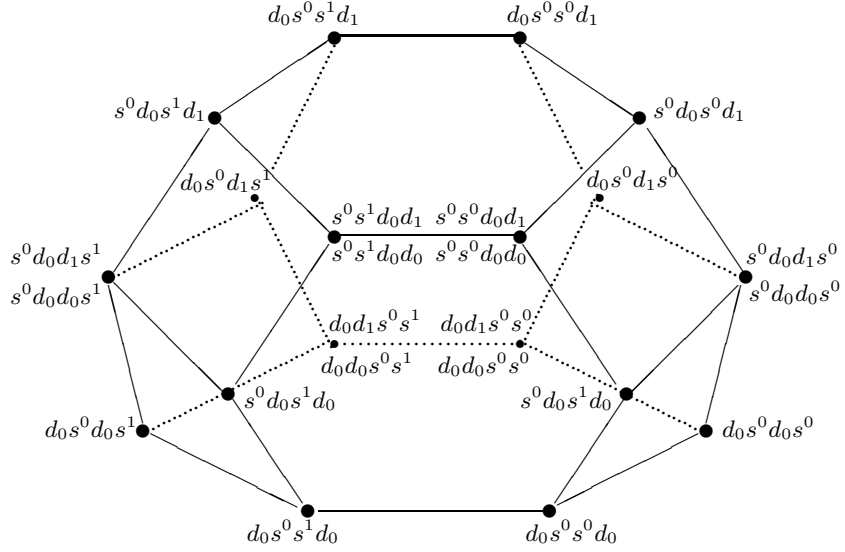
Passing to the quotient face-codegeneracy polyhedron, we see that the vertices of \mathbf{P}_n^{n+m} are now identified with factorizations of ψ of the form

(5.4)

$$E_{m+\ell}^{n+k} \xrightarrow{s^{j_{n_t}^t}} E_{m+\ell}^{n+k-1} \dots E_{m+\ell}^{n_t+1} \xrightarrow{s^{j_1^t}} E_{m+\ell}^{n_t} \xrightarrow{\theta_t} E_{m_t}^{n_t} \dots E_{m_1}^{n_1} \xrightarrow{s^{j_{n_1}^0}} \dots E_{m_1}^{n_1+1} \xrightarrow{s^{j_{n_0}^0}} E_{m_1}^n \xrightarrow{\theta_0} E_m^n$$

where θ_i is a composite of face maps (i.e., we do not distinguish the different ways of decomposing θ_i as $d_{k_1} \circ \dots \circ d_{k_r}$). The collection of such factorizations of ψ will be denoted by $D(\psi)/\sim$, where \sim is the obvious equivalence relation on $D(\psi)$. We shall denote the face-codegeneracy polyhedron \mathbf{P}_n^{n+m} with its vertices so labelled by $\mathbf{P}_n^{n+m}(\psi)$. An example for $\psi = d_0 d_1 s^0 s^1$ appears in Figure 1.

5.5. Notation. For $\psi : E_{m+\ell}^{n+k} \rightarrow E_\ell^k$ as above, we denote by $\mathcal{C}(\psi)$ the collection of all composite face-codegeneracy maps $\rho : E_{m(\rho)+\ell(\rho)}^{n(\rho)+k(\rho)} \rightarrow E_{\ell(\rho)}^{k(\rho)}$ such that ρ is of the form $\rho = \xi_t \circ \dots \circ \xi_s$ ($1 \leq s \leq t \leq \nu$) for some decomposition $\psi = \xi_\nu \circ \dots \circ \xi_1 = \theta_0 \circ s^{j_{n_0}^0} \circ \dots \circ s^{j_{n_1}^0} \circ \theta_1 \circ \dots \circ \theta_t \circ s^{j_1^t} \circ \dots \circ s^{j_{n_t}^t}$ of (5.4). That is, we allow only those subsequences $\lambda_b, \dots, \lambda_a$ of a factorization $\psi = \lambda_{n+m} \circ \dots \circ \lambda_1$ in $\mathcal{D}(\psi)$ which are compatible with the equivalence relation \sim in the sense that λ_{b+1} and λ_b are not both face maps, and similarly for λ_{a-1} and λ_a . Such a ρ will be called *allowable*.

FIGURE 1. The face-codegeneracy polyhedron $\mathbf{P}_2^4(d_0 d_1 s^0 s^1)$

5.6. Higher homotopy operations. Given a cosimplicial simplicial space up-to-homotopy ${}^h\mathbf{W}_\bullet$ as in §4.2, we now define a certain sequence of higher homotopy operations. First recall that the *half-smash* of two spaces $\mathbf{X}, \mathbf{Y} \in \mathcal{T}_*$ is $\mathbf{X} \ltimes \mathbf{Y} := (\mathbf{X} \times \mathbf{Y})/(\mathbf{X} \times \{*\})$; if \mathbf{X} is a suspension, there is a (non-canonical) homotopy equivalence $\mathbf{X} \ltimes \mathbf{Y} \simeq \mathbf{X} \wedge \mathbf{Y} \vee \mathbf{X}$.

5.7. Definition. Given a composite face-codegeneracy map $\psi : \mathbf{W}_{m+\ell}^{n+k} \rightarrow \mathbf{W}_\ell^k$ as above, a *compatible collection for $\mathcal{C}(\psi)$* and ${}^h\mathbf{W}_\bullet$ is a set $\{g^\rho\}_{\rho \in \mathcal{C}(\psi)}$ of maps $g^\rho : \mathbf{P}_{m(\rho)}^{n(\rho)+m(\rho)}(\rho) \ltimes \mathbf{W}_{m(\rho)+\ell(\rho)}^{n(\rho)+k(\rho)} \rightarrow \mathbf{W}_{\ell(\rho)}^{k(\rho)}$ for each $\rho \in \mathcal{C}(\psi)$, satisfying the following condition:

Assume that for such a $\rho \in \mathcal{C}(\psi)$ we have some decomposition

$$\rho = \xi_\nu \circ \dots \circ \xi_1 = \theta_0 \circ s_{n_0}^0 \circ \dots \circ s_{n_1}^0 \circ \theta_1 \circ \dots \circ \theta_t \circ s_{j_1}^1 \circ \dots \circ s_{n_t}^1$$

in $\mathcal{D}(\rho)/\sim$, as in (5.4), and let

$$\mathbb{P} = \langle 1, \dots, \ell_1 \mid \dots \mid \ell_{i-1} + 1, \dots, \ell_i \mid \dots \mid \ell_{r-1} + 1, \dots, \nu \rangle$$

be a partition of $(1, \dots, \nu)$ as in §5.2, yielding a sequence of composite face-codegeneracy maps $\rho_i \in \mathcal{C}(\rho) \subseteq \mathcal{C}(\psi)$ for $i = 1, \dots, r$.

Let $Q(\mathbb{P}) \cong \mathbf{P}_{n_1}^{\ell_1}(\rho_1) \times \dots \times \mathbf{P}_{n_i}^{\ell_i - \ell_{i-1}}(\rho_i) \times \dots \times \mathbf{P}_{n_r}^{\nu - \ell_{r-1}}(\rho_r)$ be the corresponding sub-polyhedron of $\mathbf{P}_{m(\rho)}^{n(\rho)+m(\rho)}(\rho)$. Then we require that $g^\rho|_{Q(\mathbb{P}) \ltimes \mathbf{W}_{m(\rho)+\ell(\rho)}^{n(\rho)+k(\rho)}}$ be the composite of the corresponding maps g^{ρ_i} in the sense that

$$(5.8) \quad g^\rho(x_1, \dots, x_r, w) = g^{\rho_1}(x_1, g^{\rho_2}(x_2, \dots, g^{\rho_r}(x_r, w) \dots))$$

for $x_i \in \mathbf{P}_{n_i}^{\ell_i - \ell_{i-1}}(\rho_i)$ and $w \in \mathbf{W}_{m(\rho)+\ell(\rho)}^{n(\rho)+k(\rho)}$.

We further require that if $\rho = \lambda_1$ is of length 1, then g^ρ must be in the prescribed homotopy class of the face or codegeneracy map λ_1 . Thus in particular, for each

vertex \bar{p}_σ of $\mathbf{P}_n^{n+m}(\psi)$, indexed by a factorization $\psi = \xi_\nu \circ \cdots \circ \xi_1$ in $\mathcal{D}(\psi)/\sim$, the map $g^\rho|_{\{\bar{p}_\sigma\} \times \mathbf{W}_{m+k}^{n+\ell}}$ represents the class $[\xi_\nu \circ \cdots \circ \xi_1]$.

5.9. Fact. Any compatible collection of maps $\{g^\rho\}_{\rho \in \mathcal{C}(\psi)}$ for $C(\psi)$ induces a map $f = f^\psi : \partial \mathbf{P}_n^{n+m} \ltimes \mathbf{W}_{m+\ell}^{n+k} \rightarrow \mathbf{W}_\ell^k$ (since all the facets of $\partial \mathbf{P}_n^{n+m}$ are products of face-codegeneracy polyhedra of the form $\mathbf{P}_{n(\rho)}^{n(\rho)+m(\rho)}(\rho)$ for $\rho \in \mathcal{C}(\psi)$, and condition (5.8) guarantees that the maps g^ρ agree on intersections).

5.10. Definition. Given ${}^h\mathbf{W}_\bullet$ as in §4.10, for each $k \geq 2$ and each composite face-codegeneracy map $\psi : \mathbf{W}_{m+\ell}^{n+k} \rightarrow \mathbf{W}_\ell^k$, the k -th order homotopy operation associated to ${}^h\mathbf{W}_\bullet$ and ψ is a subset $\langle \psi \rangle$ of the track group $[\Sigma^{n+m-2} \mathbf{W}_{m+\ell}^{n+k}, \mathbf{W}_\ell^k]$, defined as follows:

Let $S \subseteq [\partial \mathbf{P}_n^{n+m} \ltimes \mathbf{W}_{m+\ell}^{n+k}, \mathbf{W}_\ell^k]$ be the set of homotopy classes of maps $f = f^\psi : \partial \mathbf{P}_n^{n+m} \ltimes \mathbf{W}_{m+\ell}^{n+k} \rightarrow \mathbf{W}_\ell^k$ which are induced as above by some compatible collection $\{g^\rho\}_{\rho \in \mathcal{C}(\psi)}$ for $\mathcal{C}(\psi)$.

Now choose a splitting

$$(5.11) \quad \partial \mathbf{P}_n^{n+m}(\psi) \ltimes \mathbf{W}_{m+\ell}^{n+k} \cong \mathbf{S}^{n+m-2} \ltimes \mathbf{W}_{m+\ell}^{n+k} \simeq (\mathbf{S}^{n+m-2} \wedge \mathbf{W}_\ell^k) \vee \mathbf{W}_\ell^k$$

and let $\langle \psi \rangle \subseteq [\Sigma^{n+m-2} \mathbf{W}_{m+\ell}^{n+k}, \mathbf{W}_\ell^k]$ be the image of the subset S under the resulting projection.

It is clearly a necessary condition in order for the subset $\langle \psi \rangle$ to be non-empty that all the lower order operations $\langle \rho \rangle$ *vanish* (i.e., contain the null class) for all $\rho \in \mathcal{C}(\psi) \setminus \{\psi\}$ – because otherwise the various maps $g^\rho : \mathbf{P}_{m(\rho)}^{n(\rho)+m(\rho)}(\rho) \ltimes \mathbf{W}_{m(\rho)+\ell(\rho)}^{n(\rho)+k(\rho)} \rightarrow \mathbf{W}_{\ell(\rho)}^{k(\rho)}$ cannot even extend over the interior of $\mathbf{P}_{m(\rho)}^{n(\rho)+m(\rho)}(\rho)$. A *sufficient* condition is that the operations $\langle \rho \rangle$ *vanish coherently*, in the sense that the choices of compatible collections for the various ρ be consistent on common subpolyhedra (see [B5, §5.7] for the precise definition, and [B5, §5.9] for the obstructions to coherence).

On the other hand, if ${}^h\mathbf{W}_\bullet$ is the cosimplicial simplicial space up-to-homotopy of §4.4 (corresponding to the cosimplicial simplicial Π -algebra $(E_\bullet)_\Delta^\bullet$ with the CW -basis $\{\bar{E}_r^n\}_{r,n=0}^\infty$), then the vanishing of the homotopy operation $\langle \psi|_{\bar{C}_r^n} \rangle$ – with ψ restricted to the (n, r) -cross-term – implies the vanishing of $\langle \psi \rangle$, for any $\psi : \mathbf{W}_{m+\ell}^{n+k} \rightarrow \mathbf{W}_\ell^k$ (assuming lower order vanishing). This is because outside of the wedge summand \bar{C}_r^n , the map ψ is determined by the maps $\rho \in \mathcal{C}(\psi)$ and the coface and degeneracy maps of ${}^h\mathbf{W}_\bullet$, which we may assume to ∞ -homotopy commute by induction and 4.9 respectively.

We may thus sum up the results of this section, combined with Proposition 4.11, in:

5.12. Theorem. A space $\mathbf{X} \in \mathcal{T}_*$, for which $\pi_* \mathbf{X}$ is an abelian Π -algebra, is homotopy equivalent to a loop space if and only if all the higher homotopy operations $\langle \psi|_{\bar{C}_r^n} \rangle$ defined above vanish coherently.

5.13. Remark. As observed in §4.2, for any $\mathbf{X} \in \mathcal{T}_*$ the space $J\mathbf{X}$ is the colimit of the Δ -cosimplicial space $\mathbf{U}(X)_\Delta^\bullet$, and in fact the n -th stage of the James construction, $J_n \mathbf{X}$, is the (homotopy) colimit of the $(n-1)$ -coskeleton of $\mathbf{U}_\Delta^\bullet$. Thus if we think of the sequence of higher homotopy operations “in the simplicial direction” as obstructions to the validity of the identity [B7, Thm. 5.7(*)] (up to ∞ -homotopy

Since $\pi_*\mathbf{S}^7$ is a free Π -algebra, it has a very simple CW -resolution $A_\bullet \rightarrow \pi_*\mathbf{S}^7$, with $\bar{A}_0 \cong \pi_*\mathbf{S}^7$ (generated by ι^7), and $\bar{A}_r = 0$ for $r \geq 1$. A cross-term basis (§4.4) for the cosimplicial simplicial Π -algebra E_\bullet of §4.10 is then given in dimensions < 24 by:

- $\bar{C}_1^1 \cong \pi_*\mathbf{S}^{13}$, with $\bar{d}_0\iota^{13} = [d^0\iota^7, d^1\iota^7]$;
- $\bar{C}_2^2 \cong \pi_*\mathbf{S}^{19}$, with $\bar{d}_0\iota^{19} = [d^0\iota^{13}, s_0d^2d^1\iota^7] - [d^1\iota^{13}, s_0d^2d^0\iota^7] + [d^2\iota^{13}, s_0d^1d^0\iota^7]$;
- \bar{C}_r^n is at least 24-connected for all other n, r .

We set $s_r^j|_{\bar{C}_r^n} = 0$ for all $n \leq 2$; this determines E_\bullet in degrees ≤ 21 and cosimplicial dimensions ≤ 2 .

By Remark 5.13, the two secondary operations $\langle d_0s^0|_{\bar{C}_1^1} \rangle$ and $\langle d_1s^0|_{\bar{C}_1^1} \rangle$ must vanish; on the other hand, by Corollary 5.14 all obstructions to \mathbf{S}^7 being a loop space are in degrees ≥ 21 , so the only relevant cross-term is \bar{C}_2^2 , with three possible third-order operations $\langle \psi|_{\bar{C}_2^2} \rangle$, for $\psi = d_0d_1s^0s^1$, $d_0d_2s^0s^1$, or $d_1d_2s^0s^1$. The corresponding face-codegeneracy polyhedra $P_2^4(\psi)$ is as in Figure 2.

It is straightforward to verify that the operations $\langle \psi|_{\bar{C}_2^2} \rangle$ are trivial for $\psi = d_0d_2s^0s^1$ or $d_1d_2s^0s^1$ (in fact, many of the maps g^ρ , for $\rho \in C(\psi)$, may be chosen to be null). One may also show that there is a compatible collection $\{g^\rho\}_{\rho \in C(\varphi)}$ for $\varphi = d_0d_1s^0s^1$, in the sense of §5.7, so that the corresponding subset $\langle \varphi|_{\bar{C}_2^2} \rangle \subseteq \pi_{21}\mathbf{S}^7$ is non-empty; in fact, it contains the only possible obstruction to the 21-Postnikov approximation for \mathbf{S}^7 to be a loop space.

The existence of the tertiary operation $\langle \varphi|_{\bar{C}_2^2} \rangle$ corresponds to the fact that the element $[[\iota^7, \iota^7], \iota^7] - [[\iota^7, \iota^7], \iota^7] + [[\iota^7, \iota^7], \iota^7] \in \pi_{21}\mathbf{S}^7$ is trivial “for three different reasons”: because of the Jacobi identity, because all Whitehead products vanish in $\pi_*\mathbf{S}^7$, and because of the linearity of the Whitehead product – i.e., $[0, \alpha] = 0$.

On the other hand, we know that there is a 3-primary obstruction to the homotopy-associativity of any H -multiplication on \mathbf{S}^7 , namely the element $\sigma_{14}^\# \tau_7 \in \pi_{21}\mathbf{S}^7$ (see [J1, Theorem 2.6]). We deduce that $0 \notin \langle \varphi|_{\bar{C}_2^2} \rangle$, and in fact (modulo 3) this tertiary operation consists exactly of the elements $\pm \sigma_{14}^\# \tau_7$.

For a detailed calculation of such higher order operations using simplicial resolutions of Π -algebras, see [B6, §4.13].

5.16. *Remark.* Our approach to the question of whether \mathbf{X} is a loop space is clearly based on, and closely related to, the classical approaches of Sugawara and Stasheff (cf. [St1, St2, Su]). One might wonder why Stasheff’s associahedra K_i (cf. [St1, §2,6]) do not show up among the face-codegeneracy polyhedra we describe above. Apparently this is because we do not work directly with the space \mathbf{X} , but rather with its simplicial resolution, which may be thought of as a “decomposition” of \mathbf{X} into wedges of spheres.

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